

BILINEAR LOCAL SMOOTHING ESTIMATE FOR AIRY EQUATION

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ABSTRACT. In this short note, we prove a refinement of bilinear local smoothing estimate to Airy solutions, when the frequency support of two wave are separated. As an application we prove a smoothing property of a bilinear form.

The purpose of this short note is to show a refinement of bilinear local smoothing estimate for Airy solutions:

$$\partial_t u + \partial_x^3 u = 0. \quad (1)$$

We denote $e^{-t\partial_x^3} u_0$ the linear solution, i.e.

$$e^{-t\partial_x^3} u_0 = \frac{1}{2\pi} \int e^{it\xi^3 + i(x-y)\xi} u_0(y) dy d\xi.$$

The Airy equation is the linear part of the generalised Korteweg-de Vries equation:

$$\partial_t u + \partial_x^3 u + \partial_x(u^p) = 0. \quad (2)$$

Local smoothing phenomenon is due to dispersive nature of linear dispersive equation, and firstly formulated by Kato [3]. As like KdV equation, when the nonlinear term has a derivative the (local) smoothing estimate is crucial to build the Picard iteration in the well-posedness theory. Indeed, Kenig-Ponce-Vega [4] developed and used the following local smoothing estimate to obtain the well-posedness.

Proposition 1. *We have*

$$\|D^\alpha e^{-t\partial_x^3} f\|_{L_x^q L_t^r} \lesssim \|f\|_{L_x^2} \quad (3)$$

where $-\alpha + \frac{1}{q} + \frac{3}{r} = \frac{1}{2}$, and $\frac{4}{q} + \frac{2}{r} \leq 1$, except at an end point $(q, r) = (\infty, \infty)$. Here D^α is a homogeneous fractional derivative. See Notations.

In particular,

$$\|e^{-t\partial_x^3} u_0\|_{L_x^5 L_t^{10}} \lesssim \|u_0\|_{L_x^2}. \quad (4)$$

Moreover, we have the inhomogeneous local smoothing estimate.

$$\left\| \int e^{s\partial_x^3} D^\alpha F(s, x) ds \right\|_{L_x^2} \lesssim \|F\|_{L_x^{q'} L_t^{r'}} \quad (5)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$ where α, q and r as above.

In this note, we consider the interaction of two Airy wave when the support of frequencies are separated and show an improved version of bilinear local smoothing estimate.

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Theorem 1. *Let $M, N > 0$. Then,*

$$\|D^\alpha e^{-t\partial_x^3} f D^\alpha e^{-t\partial_x^3} g\|_{L_x^{q/2} L_t^{r/2}} \lesssim \left(\frac{M}{N}\right)^{5\theta/12} \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (6)$$

where

$$-\alpha + \frac{1}{q} + \frac{3}{r} = \frac{1}{2}, \left(\frac{1}{q}, \frac{1}{r}\right) = \left(\frac{\theta}{6} + \frac{1-\theta}{4}, \frac{\theta}{6}\right), 0 \leq \theta \leq 1$$

for all L^2 -functions f and g with $\text{supp } \widehat{f} \subset \{\xi : |\xi| \leq 2M\}$ and $\text{supp } \widehat{g} \subset \{\xi : N \leq |\xi| \leq 2N\}$, $0 \leq M \leq N$.

In particular, we have

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_x^{5/2} L_t^5} \lesssim \left(\frac{M}{N}\right)^{1/4} \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (7)$$

In the space-time frequency space, the linear wave is supported on the characteristic curve $\tau = \xi^3$. Due to the curvature (or the slope of the tangent line) of the interaction of two linear waves at different frequencies is weaker and so one can have some gain.

Remark 1. *This type of estimate is firstly shown by Bourgain for symmetric Strichartz estimate of Schrödinger equation in $d = 2$ [1]. Keraani-Vargas [5] extended to other dimension for symmetric Strichartz norms and Chae-Cho-Lee [2] for non-symmetric norms.*

Remark 2. *The exponent in Theorem 1 is sharp.*

Define $\widehat{f} = \chi_{M \leq \xi \leq 2M}$ and $\widehat{g} = \chi_{1 \leq \xi \leq 1+M^{1/2}}$. Consider a subset K of $\mathbb{R} \times \mathbb{R}$ of (t, x)

$$K = \{(t, x) : |x + 3t| \leq \frac{1}{100} M^{-1/2}, |x| \leq \frac{1}{100} M^{-1}\}.$$

One can easily observe that for all $(t, x) \in K$,

$$|e^{-t\partial_x^3} f(x)| = \left| \int_M^{2M} e^{it\xi^3 + ix\xi} d\xi \right| \sim M$$

and

$$|e^{-t\partial_x^3} g(x)| = \left| \int_1^{1+M^{1/2}} e^{it\xi^3 + ix\xi} d\xi \right| \sim M^{1/2}.$$

Thus,

$$\begin{aligned} \|D^\alpha e^{-t\partial_x^3} f A^\alpha e^{-t\partial_x^3} g\|_{L_x^{\frac{q}{2}} L_t^{\frac{r}{2}}} &\geq M^1 M^{\frac{1}{2}} M^\alpha \|\chi_K\|_{L_x^{q/2} L_t^{r/2}} \\ &\sim M^{\frac{3}{2} + \alpha} M^{-\frac{1}{2} \cdot \frac{2}{r}} M^{-1 \cdot \frac{2}{q}} \\ &\sim M^{1 + \frac{2}{r} - \frac{1}{q}} = M^{\frac{5\theta}{12} + \frac{3}{4}} \end{aligned}$$

where used admissible condition of exponents.

Since $\|f\|_{L_x^2} = M^{1/2}$ and $\|g\|_{L_x^2} = M^{1/4}$, we see the estimate (6) is sharp.

Proof of Theorem 1.

Since (6) is a scaling invariant estimate, by scaling one can assume $N = 1$. In view of (3), we may also assume $M \ll 1$. (6) follows by interpolating the following two estimates:

$$\|D^{-1/4} e^{-t\partial_x^3} f D^{-1/4} e^{-t\partial_x^3} g\|_{L_x^2 L_t^\infty} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (8)$$

$$\|D^{1/6} e^{-t\partial_x^3} f D^{1/6} e^{-t\partial_x^3} g\|_{L_{x,t}^3} \lesssim M^{5/12} \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (9)$$

(8) is an immediate result of (3). Now we prove (9). Using Bernstein's inequality and observing frequency support of f and g , we are reduced to show that

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^3} \lesssim M^{1/4} \|f\|_{L_x^2} \|g\|_{L_x^2}. \quad (10)$$

(10) is derived from the following lemma: indeed, it follows from the interpolation of (11)(p=2) and (12)

□

Lemma 1. *Assume that f and g are functions such that $\text{supp}|\widehat{f}| \subset [0, 2M]$, $\text{supp}|\widehat{g}| \subset [1, 2]$, $M \ll 1$*
 (a) *Let $p \geq 2$. Then we have*

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^p} \lesssim \|\widehat{f}\|_{L_\xi^{p'}} \|\widehat{g}\|_{L_\xi^{p'}} \quad (11)$$

where $p' = \frac{p}{p-1}$.

(b)

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^4} \lesssim M^{3/8} \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (12)$$

Lemma 1(a) follows from a classical argument of Fefferman and Stein [6]. It makes use of Hausdorff-Young inequality. We give a proof in the appendix for the sake of completeness.

In order to show (12) we use the example of Remark 2. Decompose g into functions whose frequency support are on small intervals of length $M^{1/2}$. Indeed, for integer k , $1 \leq k \leq M^{-1/2}$, set $I_k = [1 + (k-1)M^{1/2}, 1 + kM^{1/2}]$ and set $\widehat{g}_k = \widehat{g}\chi_{I_k}$. Then $g = \sum_k g_k$. Then we will use the following orthogonality inequality:

Lemma 2. *We have*

$$\left\| \sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right\|_{L_{t,x}^4} \lesssim \left(\sum_k \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k\|_{L_{t,x}^4}^2 \right)^{1/2} \quad (13)$$

Proof.

We write, using Plancherel theorem,

$$\begin{aligned} \left\| \sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right\|_{L_{x,t}^4}^2 &= \left\| \left(\sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right)^2 \right\|_{L_{t,x}^2} \\ &= \left\| \sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \sum_j e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j \right\|_{L_{t,x}^2} \\ &= \left\| \sum_{j,k} \widetilde{e^{-t\partial_x^3} f * e^{-t\partial_x^3} f * e^{-t\partial_x^3} g_j * e^{-t\partial_x^3} g_k} \right\|_{L_{\tau,\xi}^2}. \end{aligned}$$

where $\widetilde{f(t,x)}(\tau,\xi)$ is the space-time Fourier transform of $f(t,x)$. We denote by $E_{j,k}$ the support of the function $\widetilde{e^{-t\partial_x^3} f * e^{-t\partial_x^3} f * e^{-t\partial_x^3} g_j * e^{-t\partial_x^3} g_k}$. We claim that the $E_{j,k}$ are essentially disjoint. In other words, there is a constant C , independent of M , so that

$$\sum_{j,k} \chi_{E_{j,k}} \leq C. \quad (14)$$

By this claim, we estimate

$$\begin{aligned}
& \left\| \sum_{j,k} \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k} \right\|_{L_{t,x}^2} \\
& \leq C \left(\sum_{j,k} \left\| \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k} \right\|_{L_{\tau,\xi}^2}^2 \right)^{1/2} \\
& = C \left(\sum_{j,k} \left\| e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j \right\|_{L_{\tau,\xi}^2} \right)^{1/2} \\
& = C \left(\sum_{j,k} \int |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j|^2 \right)^{1/2} \\
& = C \left(\int \left(\sum_k |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k|^2 \right)^2 \right)^{1/2} \\
& = C \left\| \sum_k |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k|^2 \right\|_{L^2} \\
& \leq C \sum_k \left\| |e^{-t\partial_x^3} f e^{-t\partial_x^3} g|^2 \right\|_{L^2} \\
& = C \sum_k \left\| e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right\|_{L^4}^2.
\end{aligned}$$

We are left to show the inequality (14). One can easily see that the support of $\widetilde{e^{-t\partial_x^3} g_k}$ is in $E_k = \{(\tau, \xi) : |\xi - kM^{1/2}| \leq M^{1/2}, \tau = \xi^3\}$, and the support of $\widetilde{e^{-t\partial_x^3} f}$ is in $\{(\tau, \xi) : |\xi| \leq 2M, \tau = \xi^3\}$. If $(\rho, \eta) \in E_{j,k}$, then there exists (ξ_1, ξ_2) such that $(\xi_1^3, \xi_1) \in E_k, (\xi_2^3, \xi_2) \in E_j, |\rho - \xi_1^3 - \xi_2^3| \leq 4M$, and $|\eta - \xi_1 - \xi_2| \leq 4M$.

From the identity $4\xi_1^3 + 4\xi_2^3 = (\xi_1 + \xi_2)^3 + 3(\xi_1 - \xi_2)^2(\xi_1 + \xi_2)$, we see that

$$E_{j,k} \subset F_{j,k} = \{(\rho, \eta) : |\eta - (j+k)M^{1/2}| \leq 3M^{1/2}, (3|k-j|^2 - 8)M \leq |4\rho - \eta^3| \leq (6|k-j|^2 + 8)M\}.$$

It is easy to verify that the $F_{j,k}$'s overlap only a finite number of times and that this number is bounded by a universal constant. \square

Proof of (12).

In view of (13), we are reduced to show

$$\left(\sum_k \left\| e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right\|_{L_{t,x}^4}^2 \right)^{1/2} \lesssim M^{3/8} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

Using (11) for $p = 4$, and the size of support of f and g_k , we estimate

$$\begin{aligned}
\left(\sum_k \left\| e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \right\|_{L_{t,x}^4}^2 \right)^{1/2} & \lesssim \left(\sum_k \|\widehat{f}\|_{L^{4/3}}^2 \|\widehat{g_k}\|_{L^{4/3}}^2 \right)^{1/2} \\
& \lesssim M^{3/8} \|\widehat{f}\|_{L^2} \left(\sum_k \|\widehat{g_k}\|_{L^2}^2 \right)^{1/2} \\
& = M^{3/8} \|f\|_{L^2} \|g\|_{L^2}
\end{aligned}$$

which concludes (12). \square

As a corollary we can observe the smoothing property of bilinear form.

Corollary 1. *Let $\frac{3}{4} < b < 1$.*

$$\|\partial_x(e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0)\|_{L_x^{5/2}L_t^5} \lesssim \|u_0\|_{H^b}\|v_0\|_{H^{1-b}} \quad (15)$$

We have a $\frac{1}{4} - \epsilon$ derivative gain in the bilinear form compared when used (3). The proof is via the frequency decomposition. In the high-low interaction when a derivative falls to high frequency, the refined bilinear estimate helps to move some derivative to low frequency part. We add the proof in the Appendix.

APPENDIX

Proof of (11).

Writing

$$e^{-t\partial_x^3}fe^{-t\partial_x^3}g(t, x) = c \iint e^{ix(\xi_1+\xi_2)+it(\xi_1^3+\xi_2^3)} \widehat{f}(\xi_1)\widehat{g}(\xi_2) d\xi_1 d\xi_2,$$

we make a change of variables $(u, v) = (\xi_1 + \xi_2, \xi_1^3 + \xi_2^3)$. Then we obtain

$$e^{-t\partial_x^3}fe^{-t\partial_x^3}g(t, x) = c \iint e^{ixu+itv} \Pi(u, v) dudv$$

where $\Pi(u, v) = \widehat{f}(\xi_1)\widehat{g}(\xi_2)|J^{-1}|$ and $J = \det \frac{\partial(u, v)}{\partial(\xi_1, \xi_2)} = \frac{1}{3(\xi_2^2 - \xi_1^2)}$.

We can view

$$e^{-t\partial_x^3}fe^{-t\partial_x^3}g(t, x) = \widehat{\Pi}(t, x).$$

Hence, using Hausdorff-Young inequality, for $p \geq 2$,

$$\|e^{-t\partial_x^3}fe^{-t\partial_x^3}g\|_{L_{t,x}^p} = \|\widehat{\Pi}\|_{L_{t,x}^p} \leq \|\Pi\|_{L_{t,x}^{p'}}$$

where $p' = \frac{p}{p-1}$.

To compute $\|\Pi\|_{L^{p'}}$, we use the fact $|\xi_1 - \xi_2| \geq 1/2$ (i.e. $|J| \sim 1$) and change variables back to ξ_1, ξ_2 . Indeed,

$$\begin{aligned} \|\Pi\|_{L^{p'}}^{p'} &= \iint |\widehat{f}(\xi_1)\widehat{g}(\xi_2)J^{-1}|^{p'} dudv \\ &= \iint |\widehat{f}(\xi_1)\widehat{g}(\xi_2)|^{p'} |J^{-1}|^{p'} |J| d\xi_1 d\xi_2 \\ &\sim \|\widehat{f}\|_{L^{p'}}^{p'} \|\widehat{g}\|_{L^{p'}}^{p'}. \end{aligned}$$

□

Proof of Corollary 1.

We use the Littlewood-Paley operators to decompose into the paraproduct:

$$\partial_x e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}v_0 = \pi_{lh} + \pi_{hh} + \pi_{hl}$$

where

$$\begin{aligned}\pi_{lh} &= \sum_{N < M} P_N \partial_x e^{-t\partial_x^3} u_0 P_M e^{-t\partial_x^3} v_0 \\ \pi_{hh} &= \sum_{N \sim M} P_N \partial_x e^{-t\partial_x^3} u_0 P_M e^{-t\partial_x^3} v_0 \\ \pi_{hl} &= \sum_{N > M} P_N \partial_x e^{-t\partial_x^3} u_0 P_M e^{-t\partial_x^3} v_0.\end{aligned}$$

By the triangle inequality, we have

$$\|\partial_x(e^{-t\partial_x^3} u_0 e^{-t\partial_x^3} v_0)\|_{L_x^{5/2} L_t^5} \leq \|\pi_{lh}\|_{L_x^{5/2} L_t^5} + \|\pi_{hh}\|_{L_x^{5/2} L_t^5} + \|\pi_{hl}\|_{L_x^{5/2} L_t^5}.$$

We estimate term by term. For the first two terms we can use the usual local smoothing estimate since the derivative falls in the low frequency part.

$$\begin{aligned}\|\pi_{hh}\|_{L_x^{5/2} L_t^5} &\lesssim \sum_{j=-1}^{\infty} \|P_{2^j} \partial_x e^{-t\partial_x^3} u_0 P_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^{5/2} L_t^5} \\ &\lesssim \sum_{j=-1}^{\infty} \|\tilde{P}_{2^j} \partial_x e^{-t\partial_x^3} u_0\|_{L_x^5 L_t^{10}} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5 L_t^{10}} \\ &\lesssim \sum_{j=-1}^{\infty} 2^j \|\tilde{P}_{2^j} e^{-t\partial_x^3} u_0\|_{L_x^5 L_t^{10}} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5 L_t^{10}} \\ &= \sum_{j=-1}^{\infty} 2^{bj} \|\tilde{P}_{2^j} e^{-t\partial_x^3} u_0\|_{L_x^5 L_t^{10}} 2^{j(1-b)} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5 L_t^{10}} \\ &\lesssim \sum_{j=-1}^{\infty} 2^{bj} \|\tilde{P}_{2^j} u_0\|_{L^2} 2^{j(1-b)} \|\tilde{P}_{2^j} v_0\|_{L_x^2} \\ &\lesssim \|u_0\|_{H^b} \|v_0\|_{H^{1-b}}\end{aligned}$$

where $\tilde{P}_{2^j} = P_{2^{j-1}} + P_{2^j} + P_{2^{j+1}}$.

$$\begin{aligned}
\|\pi_{lh}\|_{L_x^{5/2}L_t^5} &\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} \|P_{2^k} \partial_x e^{-t\partial_x^3} u_0 P_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^{5/2}L_t^5} \\
&\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} \|\tilde{P}_{2^k} \partial_x e^{-t\partial_x^3} u_0\|_{L_x^5L_t^{10}} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} 2^k \|\tilde{P}_{2^k} e^{-t\partial_x^3} u_0\|_{L_x^5L_t^{10}} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5L_t^{10}} \\
&= \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} 2^{(1-b)(k-j)} 2^{bk} \|\tilde{P}_{2^k} e^{-t\partial_x^3} u_0\|_{L_x^5L_t^{10}} 2^{j(1-b)} \|\tilde{P}_{2^j} e^{-t\partial_x^3} v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} 2^{(1-b)(k-j)} 2^{bk} \|\tilde{P}_{2^k} u_0\|_{L^2} 2^{j(1-b)} \|\tilde{P}_{2^j} v_0\|_{L_x^2} \\
&\lesssim \sum_{i=1}^{\infty} 2^{-(1-b)i} \sum_{j \geq i} 2^{b(j-i)} \|\tilde{P}_{2^{j-i}} u_0\|_{L^2} 2^{(1-b)j} \|\tilde{P}_{2^j} v_0\|_{L^2} \\
&\lesssim \|u_0\|_{H^b} \|v_0\|_{H^{1-b}}.
\end{aligned}$$

For the last term, the high-low paraproduct, we need to use improved bilinear estimate (7).

$$\begin{aligned}
\|\pi_{hl}\|_{L_x^{5/2}L_t^5} &\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} \|P_{2^j} e^{-t\partial_x^3} \partial_x u_0 P_{2^k} e^{-t\partial_x^3} v_0\|_{L_x^{5/2}L_t^5} \\
&\lesssim \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} 2^{(k-j)/4} 2^j \|P_{2^j} u_0\|_{L^2} \|P_{2^k} v_0\|_{L^2} \\
&= \sum_{j=1}^{\infty} \sum_{k=-1}^{j-1} 2^{(k-j)(b-3/4)} 2^{jb} \|P_{2^j} u_0\|_{L^2} 2^{kb} \|P_{2^k} v_0\|_{L^2} \\
&\lesssim_b \|u_0\|_{H^b} \|v_0\|_{H^{1-b}}
\end{aligned}$$

where we used Bernstein's inequality, Cauchy-Schwartz inequality, and (7). \square

Notations. We use space-time mixed norm notation:

$$\|u(t, x)\|_{L_x^q L_t^r} := \left(\int \left(\int |u(t, x)|^r dt \right)^{q/r} dx \right)^{1/q}.$$

We denote the fractional derivative as $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and the Sobolev norm as

$$\|f\|_{H^s} = \|\langle D \rangle^s f\|_{L^2}$$

where $\langle \xi \rangle = |\xi| + 1$ and \widehat{f} is the Fourier transform of f . We use $X \lesssim Y$ to denote the estimate $X \leq CY$ where C depends only on the fixed parameters and exponents. We shall need the following Littlewood-Paley projection operators. Let $\phi(\xi)$ be a bump function, $\text{supp } \phi \in \{|\xi| \leq 2\}$ and $\phi(\xi) = 1$

on $\{|\xi| \leq 1\}$. For each dyadic number $N = 2^j, j \in \mathbb{N}$,

$$\begin{aligned}\widehat{P_N f}(\xi) &= (\phi(\xi/N) - \phi(2\xi/N))\widehat{f}(\xi) \\ \widehat{P_0 f}(\xi) &= (\phi(\xi))\widehat{f}(\xi) \\ \widehat{P_{\leq N} f}(\xi) &= \sum_{M \leq N} P_M f(\xi) \\ \widehat{P_{> N} f}(\xi) &= (I - \widehat{P_{\leq N}})f(\xi)\end{aligned}$$

and we also use a wider projection operator $\tilde{P}_N = P_{N/2} + P_N + P_{2N}$.

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